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Tight Distance-regular Graphs and the Subconstituent Algebra

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We consider a distance-regular graph Γ with diameter $D \geq 3$, intersection numbers a_i, b_i, c_i and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Let X denote the vertex set of Γ and fix $x \in X$. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by $A, E_0^*, E_1^*, \ldots, E_D^*$, where A denotes the adjacency matrix of Γ and E_i^* denotes the projection onto the ith subconstituent of Γ with respect to x. T is called the subconstituent algebra (or Terwilliger algebra) of Γ with respect to x. An irreducible T-module W is said to be *thin* whenever $\dim E_i^* W \leq 1$ for $0 \leq i \leq D$. By the *endpoint* of W we mean $\min\{i | E_i^* W \neq 0\}$. Let W denote a thin irreducible T-module with endpoint 1. Observe $E_1^* W$ is a one-dimensional eigenspace for $E_1^* A E_1^*$; let η denote the corresponding eigenvalue. We call η the *local eigenvalue* of W. It is known $\tilde{\theta}_1 \leq \eta \leq \tilde{\theta}_D$ where $\tilde{\theta}_1 = -1 - b_1(1+\theta_1)^{-1}$ and $\tilde{\theta}_D = -1 - b_1(1+\theta_D)^{-1}$. Let $n = 1$ or $n = D$ and assume $\eta = \tilde{\theta}_n$. We show the dimension of W is $D-1$. Let v denote a nonzero vector in $E_1^* W$. We show W has a basis $E_i v$ $(1 \leq i \leq D, i \neq n)$, where E_i denotes the primitive idempotent of A associated with θ_i. We show this basis is orthogonal (with respect to the Hermitean dot product) and we compute the square norm of each basis vector. We show W has a basis $E_{i+1}^* A_i v$ $(0 \leq i \leq D-2)$, where A_i denotes the ith distance matrix for Γ. We find the matrix representing A with respect to this basis. We show this basis is orthogonal and we compute the square norm of each basis vector. We find the transition matrix relating our two bases for W. For notational convenience, we say Γ is 1-*thin with respect to* x whenever every irreducible T-module with endpoint 1 is thin. Similarly, we say Γ is *tight with respect to* x whenever every irreducible T-module with endpoint 1 is thin with local eigenvalue $\tilde{\theta}_1$ or $\tilde{\theta}_D$. In [*J. Algebr. Comb.*, **12,** (2000), 163–197] Jurišić, Koolen and Terwilliger showed

$$\left(\theta_1 + \frac{k}{a_1+1}\right)\left(\theta_D + \frac{k}{a_1+1}\right) \geq -\frac{k a_1 b_1}{(a_1+1)^2}.$$

They defined Γ to be *tight* whenever Γ is nonbipartite and equality holds above. We show the following are equivalent: (i) Γ is tight; (ii) Γ is tight with respect to each vertex; (iii) Γ is tight with respect to at least one vertex. We show the following are equivalent: (i) Γ is tight; (ii) Γ is nonbipartite, $a_D = 0$, and Γ is 1-thin with respect to each vertex; (iii) Γ is nonbipartite, $a_D = 0$, and Γ is 1-thin with respect to at least one vertex.

1. INTRODUCTION

Let Γ denote a distance-regular graph with diameter $D \geq 3$, valency k, and intersection numbers a_i, b_i, c_i (see Section 2 for formal definitions). We recall the subconstituent algebra of Γ. Let X denote the vertex set of Γ and fix a 'base vertex' $x \in X$. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by $A, E_0^*, E_1^*, \ldots E_D^*$, where A denotes the adjacency matrix of Γ and E_i^* denotes the projection onto the ith subconstituent of Γ with respect to x. The algebra T is called the *subconstituent algebra* (or *Terwilliger algebra*) of Γ with respect to x [38]. Observe T has finite dimension. Moreover T is semi-simple; the reason is each of $A, E_0^*, E_1^*, \ldots, E_D^*$ is symmetric with real entries, so T is closed under the conjugate-transpose map [14, p. 157]. Since T is semi-simple, each T-module is a direct sum of irreducible T-modules. Describing the irreducible T-modules is an active area of research [5–13, 15–19, 21, 36, 38–40, 42] and the main topic of the present paper.

In this paper we are concerned with the irreducible T-modules that possess a certain property. In order to define this property we make a few observations. Let W denote an irreducible

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T-module. Then W is the direct sum of the nonzero spaces among $E_0^*W, E_1^*W, \ldots, E_D^*W$. There is a second decomposition of interest. To obtain it we make a defi ition. Let $k = \theta_0 > \theta_1 > \cdots > \theta_D$ denote the distinct eigenvalues of A, and for $0 \le i \le D$ let E_i denote the primitive idempotent of A associated with θ_i. Then W is the direct sum of the nonzero spaces among $E_0W, E_1W, \ldots, E_DW$. If the dimension of E_i^*W is at most 1 for $0 \le i \le D$ then the dimension of E_iW is at most 1 for $0 \le i \le D$ [38, Lemma 3.9]; in this case we say W is *thin*. Let W denote an irreducible T-module. By the *endpoint* of W we mean $\min\{i | 0 \le i \le D, E_i^*W \ne 0\}$. There exists a unique irreducible T-module with endpoint 0 [17, Proposition 8.4]. We call this module V_0. The module V_0 is thin; in fact $E_i^*V_0$ and E_iV_0 have dimension 1 for $0 \le i \le D$ [38, Lemma 3.6]. For a detailed description of V_0 see [8, 17]. In this paper we are concerned with the thin irreducible T-modules with endpoint 1.

In order to describe the thin irreducible T-modules with endpoint 1 we def ne some parameters. Let $\Delta = \Delta(x)$ denote the vertex-subgraph of Γ induced on the set of vertices in X adjacent x. The graph Δ has k vertices and is regular with valency a_1. Let $\eta_1 \ge \eta_2 \ge \cdots \ge \eta_k$ denote the eigenvalues for the adjacency matrix of Δ. We call $\eta_1, \eta_2, \ldots, \eta_k$ the *local eigenvalues* of Γ with respect to x. We mentioned Δ is regular with valency a_1 so $\eta_1 = a_1$ and $\eta_k \ge -a_1$ [2, Proposition 3.1]. It is shown in [37, Theorem 1] that $\tilde{\theta}_1 \le \eta_i \le \tilde{\theta}_D$ for $2 \le i \le k$, where $\tilde{\theta}_1 = -1 - b_1(1 + \theta_1)^{-1}$ and $\tilde{\theta}_D = -1 - b_1(1 + \theta_D)^{-1}$. We remark $\tilde{\theta}_1 < -1$ and $\tilde{\theta}_D \ge 0$, since $\theta_1 > -1$ and $a_1 - k \le \theta_D < -1$ [26, Lemma 2.6]. Let W denote a thin irreducible T-module with endpoint 1. Observe E_1^*W is a one-dimensional eigenspace for $E_1^*AE_1^*$; let η denote the corresponding eigenvalue. It turns out η is one of $\eta_2, \eta_3, \ldots, \eta_k$ so $\tilde{\theta}_1 \le \eta \le \tilde{\theta}_D$. We call η the *local eigenvalue* of W.

Let W denote a thin irreducible T-module with endpoint 1 and local eigenvalue η. To describe the structure of W we distinguish two cases: (i) $\eta = \tilde{\theta}_1$ or $\eta = \tilde{\theta}_D$; (ii) $\tilde{\theta}_1 < \eta < \tilde{\theta}_D$. We investigate case (i) in the present paper. We will investigate case (ii) in a future paper.

Concerning case (i) above, we summarize our results as follows. Let $n = 1$ or $n = D$ and def ne $\eta = \tilde{\theta}_n$. Let W denote a thin irreducible T-module with endpoint 1 and local eigenvalue η. We show the dimension of W is $D - 1$. Let v denote a nonzero vector in E_1^*W. We show W has a basis E_iv $(1 \le i \le D, i \ne n)$. We show this basis is orthogonal (with respect to the Hermitean dot product) and we compute the square norm of each basis vector. We show W has a basis $E_{i+1}^*A_iv$ $(0 \le i \le D - 2)$, where A_i denotes the ith distance matrix for Γ. We fi d the matrix representing A with respect to this basis. We show this basis is orthogonal and we compute the square norm of each basis vector. We f nd the transition matrix relating our two bases for W. Let W' denote an irreducible T-module. We show W' and W are isomorphic as T-modules if and only if W' is thin with endpoint 1 and local eigenvalue η. We show the following scalars are equal: (i) the multiplicity with which W appears in the standard module $\mathbb{C}^X$; (ii) the number of times η appears among $\eta_2, \eta_3, \ldots, \eta_k$.

For notational convenience, we say Γ is 1-*thin with respect to* x whenever every irreducible T-module with endpoint 1 is thin. Similarly, we say Γ is *tight with respect to* x whenever every irreducible T-module with endpoint 1 is thin with local eigenvalue $\tilde{\theta}_1$ or $\tilde{\theta}_D$.

In [26] Jurišić *et al.* showed

$$\left(\theta_1 + \frac{k}{a_1 + 1}\right)\left(\theta_D + \frac{k}{a_1 + 1}\right) \ge -\frac{ka_1b_1}{(a_1 + 1)^2}. \tag{1}$$

They def ned Γ to be *tight* whenever Γ is nonbipartite and equality holds in (1). We obtain the following two characterizations of the tight condition in terms of the subconstituent algebra. We show the following are equivalent: (i) Γ is tight; (ii) Γ is tight with respect to each vertex; (iii) Γ is tight with respect to at least one vertex. We show the following are equivalent:

(i) Γ is tight; (ii) Γ is nonbipartite, $a_D = 0$, and Γ is 1-thin with respect to each vertex; (iii) Γ is nonbipartite, $a_D = 0$, and Γ is 1-thin with respect to at least one vertex.

For more information on the tight condition and related topics we refer the reader to [22–25, 27–35, 41].

2. Preliminaries Concerning Distance-regular Graphs

In this section we review some defi itions and basic concepts concerning distance-regular graphs. For more background information we refer the reader to [1, 3, 20] or [38].

Let X denote a nonempty f nite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We endow V with the Hermitean inner product $\langle , \rangle$ def ned by

$$\langle u, v\rangle = u^t \overline{v} \qquad (u, v \in V), \tag{2}$$

where t denotes transpose and $-$ denotes complex conjugation. As usual, we abbreviate $\|u\|^2 = \langle u, u\rangle$ for all $u \in V$. For all $y \in X$, let $\hat{y}$ denote the element of V with a 1 in the y coordinate and 0 in all other coordinates. We observe $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V. The following formula will be useful. For all $B \in \mathrm{Mat}_X(\mathbb{C})$ and for all $u, v \in V$,

$$\langle Bu, v\rangle = \langle u, \overline{B}^t v\rangle. \tag{3}$$

Let $\Gamma = (X, R)$ denote a finite undirected, connected graph, without loops or multiple edges, with vertex set X and edge set R. Let ∂ denote the path-length distance function for Γ, and set

$$D = \max\{\partial(x, y) \mid x, y \in X\}.$$

We refer to D as the *diameter* of Γ. Let x, y denote vertices of Γ. We say x, y are *adjacent* whenever xy is an edge. Let k denote a nonnegative integer. We say Γ is *regular* with *valency* k whenever each vertex of Γ is adjacent to exactly k distinct vertices of Γ. We say Γ is *distance-regular* whenever for all integers $h, i, j\,(0 \le h, i, j \le D)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\{z \in X \mid \partial(x, z) = i, \partial(z, y) = j\}| \tag{4}$$

is independent of x and y. The integers p^h_{ij} are called the *intersection numbers* of Γ. We abbreviate $c_i = p^i_{1i-1}\,(1 \le i \le D)$, $a_i = p^i_{1i}\,(0 \le i \le D)$, and $b_i = p^i_{1i+1}\,(0 \le i \le D-1)$. For notational convenience, we def ne $c_0 = 0$ and $b_D = 0$. We note $a_0 = 0$ and $c_1 = 1$.

For the rest of this paper we assume Γ is distance-regular with diameter $D \ge 3$.

By (4) and the triangle inequality,

$$p^h_{1j} = 0 \qquad \text{if } |h - j| > 1 \qquad (0 \le h, j \le D). \tag{5}$$

Observe Γ is regular with valency $k = b_0$, and that

$$c_i + a_i + b_i = k \qquad (0 \le i \le D). \tag{6}$$

Moreover $b_i > 0$ $(0 \le i \le D-1)$ and $c_i > 0$ $(1 \le i \le D)$. For $0 \le i \le D$ we abbreviate $k_i = p^0_{ii}$, and observe

$$k_i = |\{z \in X \mid \partial(x, z) = i\}|, \tag{7}$$

where x is any vertex in X. Apparently $k_0 = 1$ and $k_1 = k$. By [1, p. 195] we have

$$k_i = \frac{b_0 b_1 \ldots b_{i-1}}{c_1 c_2 \ldots c_i} \qquad (0 \leq i \leq D). \tag{8}$$

We recall the Bose–Mesner algebra of Γ. For $0 \leq i \leq D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with xy entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x, y) = i \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \qquad (x, y \in X).$$

We call A_i the ith *distance matrix* of Γ. For convenience we defin $A_i = 0$ for $i < 0$ and $i > D$. We abbreviate $A = A_1$ and call this the *adjacency* matrix of Γ. We observe

$$\begin{aligned}
A_0 &= I, && && (9)\\
\sum_{i=0}^{D} A_i &= J, && && (10)\\
\overline{A}_i &= A_i && (0 \leq i \leq D), && (11)\\
A_i^t &= A_i && (0 \leq i \leq D), &&\\
A_i A_j &= \sum_{h=0}^{D} p_{ij}^h A_h && (0 \leq i, j \leq D), && (12)
\end{aligned}$$

where I denotes the identity matrix and J denotes the all 1's matrix. Let M denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A. We refer to M as the *Bose–Mesner algebra* of Γ. Using (9) and (12) one can readily show $A_0, A_1, \ldots, A_D$ form a basis for M. By [3, p. 45] M has a second basis $E_0, E_1, \ldots, E_D$ such that

$$\begin{aligned}
E_0 &= |X|^{-1} J, && && (13)\\
\sum_{i=0}^{D} E_i &= I, && && (14)\\
\overline{E}_i &= E_i && (0 \leq i \leq D), && (15)\\
E_i^t &= E_i && (0 \leq i \leq D), && (16)\\
E_i E_j &= \delta_{ij} E_i && (0 \leq i, j \leq D). && (17)
\end{aligned}$$

We refer to $E_0, E_1, \ldots, E_D$ as the *primitive idempotents* of Γ. We call E_0 the *trivial idempotent* of Γ.

We recall the eigenvalues of Γ. Since $E_0, E_1, \ldots, E_D$ form a basis for M, there exist complex scalars $\theta_0, \theta_1, \ldots, \theta_D$ such that $A = \sum_{i=0}^{D} \theta_i E_i$. Combining this with (17) we fi d $AE_i = E_i A = \theta_i E_i$ for $0 \leq i \leq D$. Using (11) and (15) we f nd $\theta_0, \theta_1, \ldots, \theta_D$ are in $\mathbb{R}$. Observe $\theta_0, \theta_1, \ldots, \theta_D$ are distinct since A generates M. By [2, Proposition 3.1] we have $\theta_0 = k$ and $-k \leq \theta_i \leq k$ for $0 \leq i \leq D$. Throughout this paper we assume $E_0, E_1, \ldots, E_D$ are indexed so that $\theta_0 > \theta_1 > \cdots > \theta_D$. We refer to θ_i as the *eigenvalue* of Γ associated with E_i. We call θ_0 the *trivial eigenvalue* of Γ. For $0 \leq i \leq D$ let m_i denote the rank of E_i. We refer to m_i as the *multiplicity* of E_i (or θ_i). From (13) we f nd $m_0 = 1$. Using (14)–(17) we fi d

$$V = E_0 V + E_1 V + \cdots + E_D V \qquad \text{(orthogonal direct sum).} \tag{18}$$

For $0 \leq i \leq D$ the space $E_i V$ is the eigenspace of A associated with θ_i. We observe the dimension of $E_i V$ is m_i.

We now recall the dual eigenvalues of Γ. Let θ denote an eigenvalue of Γ and let E denote the associated primitive idempotent. Since $A_0, A_1, \ldots, A_D$ is a basis for M, there exist complex scalars $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ such that

$$E = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i. \tag{19}$$

Evaluating (19) using (11) and (15) we see $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ are in $\mathbb{R}$. We refer to θ_i^* as the ith *dual eigenvalue* of Γ with respect to E (or θ). We call $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ the *dual eigenvalue sequence* associated with E (or θ). By [3, p. 128] we have

$$c_i\theta_{i-1}^* + a_i\theta_i^* + b_i\theta_{i+1}^* = \theta\theta_i^* \qquad (0 \leq i \leq D), \tag{20}$$

where $\theta_{-1}^*, \theta_{D+1}^*$ are indeterminates. We remark by [1, p. 62] that $\theta_0^* = m_i$ where $\theta = \theta_i$.

The eigenvalues θ_1, θ_D are of special interest to us. We mention some results on these eigenvalues that we will use later.

LEMMA 2.1 ([20, p. 264]). *Let Γ denote a distance-regular graph with diameter $D \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. Let θ denote one of θ_1, θ_D and let $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ denote the associated dual eigenvalue sequence.*

(i) *Suppose $\theta = \theta_1$. Then $\theta_0^* > \theta_1^* > \cdots > \theta_D^*$.*
(ii) *Suppose $\theta = \theta_D$. Then $(-1)^i\theta_i^* > 0$ for $0 \leq i \leq D$.*

LEMMA 2.2 ([26, LEMMA 2.6]). *Let Γ denote a distance-regular graph with diameter $D \geq 3$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Then*

(i) $-1 < \theta_1 < k$.
(ii) $a_1 - k \leq \theta_D < -1$.

Later in this paper we will discuss polynomials in one variable. We will use the following notation. We let λ denote an indeterminate. We let $\mathbb{R}[\lambda]$ denote the $\mathbb{R}$-algebra consisting of all polynomials in λ that have coefficient in $\mathbb{R}$.

3. TWO FAMILIES OF POLYNOMIALS

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$. In this section we recall two types of polynomials associated with Γ. To motivate things, we recall by (5) and (12) that

$$AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \qquad (0 \leq i \leq D), \tag{21}$$

where $b_{-1} = 0$ and $c_{D+1} = 0$. Let $f_0, f_1, \ldots, f_D$ denote the polynomials in $\mathbb{R}[\lambda]$ satisfying $f_0 = 1$ and

$$\lambda f_i = b_{i-1}f_{i-1} + a_if_i + c_{i+1}f_{i+1} \qquad (0 \leq i \leq D-1), \tag{22}$$

where $f_{-1} = 0$. Let i denote an integer $(0 \leq i \leq D)$. The polynomial f_i has degree i, and the coeff cient of λ^i is $(c_1c_2 \ldots c_i)^{-1}$. Comparing (21) and (22) we f nd $f_i(A) = A_i$. By [1, p. 63] the polynomials $f_0, f_1, \ldots, f_D$ satisfy the orthogonality relation

$$\sum_{h=0}^{D} f_i(\theta_h)f_j(\theta_h)m_h = \delta_{ij}|X|k_i \qquad (0 \leq i, j \leq D). \tag{23}$$

let θ denote an eigenvalue of Γ and let $\theta^*_0, \theta^*_1, \ldots, \theta^*_D$ denote the associated dual eigenvalue sequence. Comparing (20) and (22) using (8) we routinely obtain

$$f_i(\theta) = k_i \theta^*_i / \theta^*_0 \qquad (0 \leq i \leq D). \tag{24}$$

We remark on a special case. Setting $i = 0$ in (22) we routinely find $f_1 = \lambda$. Now setting $i = 1$ in (24) we find

$$\theta / k = \theta^*_1 / \theta^*_0. \tag{25}$$

We now recall some polynomials related to the f_i. Let $p_0, p_1, \ldots, p_D$ denote the polynomials in $\mathbb{R}[\lambda]$ satisfying

$$p_i = f_0 + f_1 + \cdots + f_i \qquad (0 \leq i \leq D). \tag{26}$$

Let i denote an integer ($0 \leq i \leq D$). The polynomial p_i has degree i, and the coefficient of λ^i is $(c_1 c_2 \ldots c_i)^{-1}$. Moreover $p_i(A) = A_0 + A_1 + \cdots + A_i$. Setting $i = D$ in this and using (10) we find $p_D(A) = J$.

A bit later we find a three-term recurrence satisfied by the polynomials p_i. To obtain it we use the following result.

LEMMA 3.1. *Let Γ denote a distance-regular graph with diameter $D \geq 3$. Let the polynomials $f_0, f_1, \ldots, f_D$ be from* (22)*, and let the polynomials $p_0, p_1, \ldots, p_D$ be from* (26)*. Then*

(i) $p_i - p_{i-1} = f_i \qquad (1 \leq i \leq D)$,
(ii) $(k - \lambda) p_i = b_i f_i - c_{i+1} f_{i+1}$ $(0 \leq i \leq D - 1)$.

PROOF. (i) Immediate from (26).

(ii) To see that the two sides are equal, in the expression on the left eliminate p_i using (26), and evaluate the result using (22) and (6). □

THEOREM 3.2. *Let Γ denote a distance-regular graph with diameter $D \geq 3$. Let the polynomials $p_0, p_1, \ldots, p_D$ be as in* (26)*. Then $p_0 = 1$ and*

$$\lambda p_i = c_{i+1} p_{i+1} + (a_i - c_{i+1} + c_i) p_i + b_i p_{i-1} \qquad (0 \leq i \leq D - 1),$$

where $p_{-1} = 0$.

PROOF. Eliminate f_i, f_{i+1} in Lemma 3.1(ii) using Lemma 3.1(i) and simplify the result using (6). □

The polynomials p_i satisfy the following orthogonality relation.

THEOREM 3.3. *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Let the polynomials $p_0, p_1, \ldots, p_D$ be as in* (26)*. Then $p_D(\theta_h) = 0$ for $1 \leq h \leq D$. Moreover*

$$\sum_{h=1}^{D} p_i(\theta_h) p_j(\theta_h)(k - \theta_h) m_h = \delta_{ij} |X| k_j b_j \qquad (0 \leq i, j \leq D - 1). \tag{27}$$

(*We recall m_h denotes the multiplicity of θ_h for* $0 \leq h \leq D$.)

PROOF. We first show $p_D(\theta_h) = 0$ for $1 \le h \le D$. Let h be given. We mentioned earlier that $p_D(A) = J$. Multiplying both sides of this equation by E_h, and recalling J is a scalar multiple of E_0, we find $p_D(A)E_h = 0$ in view of (17). Observe $p_D(A)E_h = p_D(\theta_h)E_h$ so $p_D(\theta_h) = 0$. Concerning (27), let the integers i, j be given. Without loss of generality, we may assume $i \le j$. By (26), Lemma 3.1(ii), and since $k = \theta_0$, the left-hand side of (27) is equal to

$$\sum_{h=0}^{D}(f_0(\theta_h) + f_1(\theta_h) + \cdots + f_i(\theta_h))(b_j f_j(\theta_h) - c_{j+1} f_{j+1}(\theta_h))m_h. \tag{28}$$

Evaluating (28) using (23) and recalling $i \le j$, we find (28) is equal to the right-hand side of (27). The result follows. □

The following fact will be useful.

LEMMA 3.4. *Let Γ denote a distance-regular graph with diameter $D \ge 3$. Let θ denote a nontrivial eigenvalue of Γ and let $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ denote the associated dual eigenvalue sequence. Then*

$$p_i(\theta) = \frac{b_1 b_2 \ldots b_i}{c_1 c_2 \ldots c_i} \frac{\theta_i^* - \theta_{i+1}^*}{\theta_0^* - \theta_1^*} \qquad (0 \le i \le D-1), \tag{29}$$

where the polynomials p_i are from (26). *We remark $\theta_0^* \ne \theta_1^*$ by* (25) *and since $\theta \ne k$.*

PROOF. Set $\lambda = \theta$ in Lemma 3.1(ii), and simplify the result using (8), (24), and (25). □

COROLLARY 3.5. *Let Γ denote a distance-regular graph with diameter $D \ge 3$. Let θ denote a nontrivial eigenvalue of Γ. Then*

$$1 + \theta = b_1 \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_1^*}, \tag{30}$$

where θ_0^*, θ_1^*, θ_2^* are dual eigenvalues associated with θ.

PROOF. Set $i = 1$ in (29) and observe $p_1 = 1 + \lambda$. □

4. A THIRD FAMILY OF POLYNOMIALS

Let Γ denote a distance-regular graph with diameter $D \ge 3$. In this section we use Γ to define a family of polynomials in one variable. We call these polynomials the g_i.

DEFINITION 4.1. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \ge 3$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. We fix $n = 1$ or $n = D$ and let $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ denote the dual eigenvalue sequence for θ_n. For $0 \le i \le D-1$ we define the polynomial $g_i \in \mathbb{R}[\lambda]$ by

$$g_i = \sum_{h=0}^{i} \frac{\theta_h^* - \theta_{h+1}^*}{\theta_i^* - \theta_{i+1}^*} p_h, \tag{31}$$

where the p_h are from (26). We remark the denominators in (31) are nonzero by Lemma 2.1. We emphasize g_i depends on n as well as the intersection numbers of Γ.

LEMMA 4.2. *With reference to Definition* 4.1 *we have*

$$p_i = g_i - \frac{\theta^*_{i-1} - \theta^*_i}{\theta^*_i - \theta^*_{i+1}} g_{i-1} \qquad (1 \le i \le D-1). \tag{32}$$

PROOF. Immediate from Definition 4.1. □

LEMMA 4.3. *With reference to Definition* 4.1, *the following* (*i*), (*ii*) *hold for* $0 \le i \le D-1$.

(*i*) *The degree of* g_i *is* i.
(*ii*) *The coefficient of* λ^i *in* g_i *is* $(c_1 c_2 \ldots c_i)^{-1}$.

PROOF. Routine. □

Our next goal is to give a three-term recurrence satisfied by the g_i. To do this we need the following result.

LEMMA 4.4. *With reference to Definition* 4.1,

$$(\theta_n - \lambda) g_i = b_{i+1} \frac{\theta^*_{i+1} - \theta^*_{i+2}}{\theta^*_i - \theta^*_{i+1}} p_i - c_{i+1} p_{i+1} \qquad (0 \le i \le D-2). \tag{33}$$

PROOF. To see that the two sides are equal, in the expression on the left eliminate g_i using Definition 4.1 and evaluate the result using Theorem 3.2 and (20). □

To describe the three-term recurrence satisfied by the polynomials g_i, we will need the following scalars.

DEFINITION 4.5. With reference to Definition 4.1 we define

$$\alpha_i = a_i - c_{i+1} \frac{\theta^*_i - \theta^*_{i+2}}{\theta^*_{i+1} - \theta^*_{i+2}} + c_i \frac{\theta^*_{i-1} - \theta^*_{i+1}}{\theta^*_i - \theta^*_{i+1}} \qquad (0 \le i \le D-2),$$

$$\beta_i = b_{i+1} \frac{(\theta^*_{i-1} - \theta^*_i)(\theta^*_{i+1} - \theta^*_{i+2})}{(\theta^*_i - \theta^*_{i+1})^2} \qquad (1 \le i \le D-2).$$

For notational convenience we define $\beta_0 = 0$.

THEOREM 4.6. *With reference to Definition* 4.1 *we have*

$$\lambda g_i = c_{i+1} g_{i+1} + \alpha_i g_i + \beta_i g_{i-1} \qquad (0 \le i \le D-2), \tag{34}$$

where $g_{-1} = 0$ *and the* α_i, β_i *are from Definition* 4.5.

PROOF. In (33) eliminate p_i, p_{i+1} using (32) and simplify the result using (20). □

Our next goal is to obtain an orthogonality relation satisfied by the polynomials g_i. We will use the following scalars.

DEFINITION 4.7. With reference to Definition 4.1, for $0 \le i \le D$ we define

$$w_i = \frac{m_i (k - \theta_i)(\theta_n - \theta_i)}{k|X|(\theta_n + 1)}, \tag{35}$$

where m_i denotes the multiplicity of θ_i. (The denominators in (35) are nonzero by Lemma 2.2.) We remark $w_i = 0$ if $i \in \{0, n\}$ and $w_i > 0$ if $i \notin \{0, n\}$.

THEOREM 4.8. *With reference to Definition* 4.1*, we have*

$$\sum_{h=0}^{D} g_i(\theta_h)g_j(\theta_h)w_h = \delta_{ij}\frac{k_{i+1}c_{i+1}b_{i+1}}{kb_1}\frac{\theta^*_{i+1}-\theta^*_{i+2}}{\theta^*_i-\theta^*_{i+1}}\frac{\theta^*_0-\theta^*_1}{\theta^*_1-\theta^*_2} \tag{36}$$

for $0 \le i, j \le D-2$*, where* $w_0, w_1, \ldots, w_D$ *are from Definition* 4.7*.*

PROOF. Let the integers i, j be given. Without loss of generality, we may assume $i \le j$. In the sum on the left in (36), eliminate $g_i(\theta_h)$ using Definition 4.1 and eliminate $(\theta_n - \theta_h)g_j(\theta_h)$ using Lemma 4.4. Evaluate the result using Theorem 3.3 and Corollary 3.5 (with $\theta = \theta_n$). □

5. THE SUBCONSTITUENT ALGEBRA AND ITS MODULES

In this section we recall some definitions and basic concepts concerning the subconstituent algebra and its modules. For more information we refer the reader to [5, 8, 9, 19, 21, 38].

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \ge 3$. We recall the dual Bose–Mesner algebra of Γ. For the rest of this section, fix a vertex $x \in X$. For $0 \le i \le D$ let $E^*_i = E^*_i(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with yy entry

$$(E^*_i)_{yy} = \begin{cases} 1, & \text{if} \quad \partial(x, y) = i \\ 0, & \text{if} \quad \partial(x, y) \ne i \end{cases} \qquad (y \in X). \tag{37}$$

We call E^*_i the ith dual idempotent of Γ with respect to x. We observe

$$\sum_{i=0}^{D} E^*_i = I, \tag{38}$$

$$\overline{E^*_i} = E^*_i \qquad (0 \le i \le D), \tag{39}$$

$$E^{*t}_i = E^*_i \qquad (0 \le i \le D), \tag{40}$$

$$E^*_i E^*_j = \delta_{ij} E^*_i \qquad (0 \le i, j \le D). \tag{41}$$

Using (38) and (41) we find $E^*_0, E^*_1, \ldots, E^*_D$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call M^* the *dual Bose–Mesner algebra of* Γ *with respect to* x. We recall the subconstituents of Γ. Using (37) we find

$$E^*_i V = \mathrm{Span}\{\hat{y} \mid y \in X, \partial(x, y) = i\} \qquad (0 \le i \le D). \tag{42}$$

By (42) and since $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V we find

$$V = E^*_0 V + E^*_1 V + \cdots + E^*_D V \qquad \text{(orthogonal direct sum)}. \tag{43}$$

Combining (42) and (7) we find

$$\dim E^*_i V = k_i \qquad (0 \le i \le D). \tag{44}$$

We call $E^*_i V$ the *ith subconstituent of* Γ *with respect to* x.

We recall how M and M^* are related. By [38, Lemma 3.2],

$$E^*_h A_i E^*_j = 0 \quad \text{if and only if } p^h_{ij} = 0 \qquad (0 \le h, i, j \le D). \tag{45}$$

Combining (45) and (5) we find

$$E^*_i A_j E^*_1 = 0 \qquad \text{if } |i - j| > 1 \qquad (0 \le i, j \le D). \tag{46}$$

Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*. We call T the *subconstituent algebra* of Γ *with respect to* x [38]. We observe T has finite dimension. Moreover T is semi-simple; the reason is that T is closed under the conjugate-transpose map [14, p. 157].

We now consider the modules for T. By a T-module we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$. We refer to V itself as the *standard module* for T. Let W denote a T-module. Then W is said to be *irreducible* whenever W is nonzero and W contains no T-modules other than 0 and W. Let W, W' denote T-modules. By an *isomorphism of T-modules* from W to W' we mean an isomorphism of vector spaces $\sigma : W \to W'$ such that

$$(\sigma B - B\sigma)W = 0 \qquad \text{for all } B \in T.$$

The modules W, W' are said to be *isomorphic as T-modules* whenever there exists an isomorphism of T-modules from W to W'.

Let W denote a T-module and let W' denote a T-module contained in W. Using (3) we find the orthogonal complement of W' in W is a T-module. It follows that each T-module is an orthogonal direct sum of irreducible T-modules. We mention any two nonisomorphic irreducible T-modules are orthogonal [8, Lemma 3.3].

Let W denote an irreducible T-module. Using (38)–(41) we find W is the direct sum of the nonzero spaces among $E_0^*W, E_1^*W, \ldots, E_D^*W$. Similarly using (14)–(17) we find W is the direct sum of the nonzero spaces among $E_0W, E_1W, \ldots, E_DW$. If the dimension of E_i^*W is at most 1 for $0 \le i \le D$ then the dimension of E_iW is at most 1 for $0 \le i \le D$ [38, Lemma 3.9]; in this case we say W is *thin*. Let W denote an irreducible T-module. By the *endpoint* of W we mean

$$\min\{i \mid 0 \le i \le D, E_i^*W \ne 0\}.$$

We adopt the following notational convention.

DEFINITION 5.1. For the rest of this paper we let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \ge 3$, valency k, intersection numbers a_i, b_i, c_i, Bose–Mesner algebra M, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. For $0 \le i \le D$ we let E_i denote the primitive idempotent of Γ associated with θ_i. We let V denote the standard module for Γ. We fix $x \in X$ and abbreviate $E_i^* = E_i^*(x)(0 \le i \le D)$, $M^* = M^*(x)$, $T = T(x)$. We define

$$s_i = \sum_{\substack{y \in X \\ \partial(x,y)=i}} \hat{y} \qquad (0 \le i \le D). \tag{47}$$

6. THE T-MODULE V_0

With reference to Definition 5.1, there exists a unique irreducible T-module with endpoint 0 [17, Proposition 8.4]. We call this module V_0. The module V_0 is described in [8, 17]. We summarize some details below in order to motivate the results that follow.

The module V_0 is thin. In fact each of $E_iV_0, E_i^*V_0$ has dimension 1 for $0 \le i \le D$. We give two bases for V_0. The vectors

$$E_0\hat{x}, E_1\hat{x}, \ldots, E_D\hat{x} \tag{48}$$

form a basis for V_0. The vectors in (48) are mutually orthogonal and

$$\|E_i\hat{x}\|^2 = m_i|X|^{-1} \qquad (0 \le i \le D).$$

To motivate the second basis we make some comments. For $0 \leq i \leq D$ we have $s_i = A_i\hat{x}$, where s_i is from (47). Moreover $s_i = E_i^*\delta$, where $\delta = \sum_{y\in X}\hat{y}$. The vectors

$$s_0, s_1, \ldots, s_D \tag{49}$$

form a basis for V_0. The vectors in (49) are mutually orthogonal and

$$\|s_i\|^2 = k_i \qquad (0 \leq i \leq D). \tag{50}$$

With respect to the basis given in (49) the matrix representing A is

$$\begin{pmatrix} a_0 & b_0 & & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{D-1} \\ \mathbf{0} & & & & c_D & a_D \end{pmatrix}. \tag{51}$$

The bases for V_0 given in (48), (49) are related as follows. For $0 \leq i \leq D$ we have

$$s_i = \sum_{h=0}^{D} f_i(\theta_h)E_h\hat{x},$$

where the f_i are from (22).

7. The Local Eigenvalues

Later in the paper we will consider the thin irreducible T-modules with endpoint 1. In order to discuss these we recall some parameters known as the local eigenvalues.

Definition 7.1. With reference to Defi ition 5.1, we let $\Delta = \Delta(x)$ denote the graph $(\breve{X}, \breve{R})$, where

$$\breve{X} = \{y \in X | \partial(x, y) = 1\},$$
$$\breve{R} = \{yz | y, z \in \breve{X}, yz \in R\}.$$

We observe Δ is the vertex-subgraph of Γ induced on the set of vertices in X adjacent x. The graph Δ has exactly k vertices, where k is the valency of Γ. Also, Δ is regular with valency a_1. We let $\breve{A}$ denote the adjacency matrix of Δ. The matrix $\breve{A}$ is symmetric with real entries; therefore $\breve{A}$ is diagonalizable with all eigenvalues real. We let $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_k$ denote the eigenvalues of $\breve{A}$. We mentioned Δ is regular with valency a_1 so $\eta_1 = a_1$ and $\eta_k \geq -a_1$ [2, Proposition 3.1]. We call $\eta_1, \eta_2, \ldots, \eta_k$ the local eigenvalues of Γ with respect to x.

With reference to Defi ition 5.1, we consider the f rst subconstituent E_1^*V. By (44) the dimension of E_1^*V is k. Observe E_1^*V is invariant under the action of $E_1^*AE_1^*$. To illuminate this action we make an observation. For an appropriate ordering of the vertices of Γ we have

$$E_1^*AE_1^* = \begin{pmatrix} \breve{A} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\breve{A}$ is from Defi ition 7.1. Apparently the action of $E_1^*AE_1^*$ on E_1^*V is essentially the adjacency map for Δ. In particular the action of $E_1^*AE_1^*$ on E_1^*V is diagonalizable with eigenvalues $\eta_1, \eta_2, \ldots, \eta_k$. We observe the vector s_1 from (47) is contained in E_1^*V. Using (51) we fi d s_1 is an eigenvector for $E_1^* A E_1^*$ with eigenvalue a_1. Let v denote a vector in $E_1^* V$. We observe the following are equivalent: (i) v is orthogonal to s_1; (ii) $Jv = 0$; (iii) $E_0 v = 0$; (iv) $E_0^* A v = 0$. Let U denote the orthogonal complement of s_1 in $E_1^* V$. We observe U has dimension $k-1$. Using (3) we fi d U is invariant under $E_1^* A E_1^*$. Apparently the restriction of $E_1^* A E_1^*$ to U is diagonalizable with eigenvalues $\eta_2, \eta_3, \ldots, \eta_k$. For $\eta \in \mathbb{R}$ let U_η denote the set consisting of those vectors in U that are eigenvectors for $E_1^* A E_1^*$ with eigenvalue η. We observe U_η is a subspace of U. We observe the following are equal: (i) the dimension of U_η; (ii) the number of times η appears among $\eta_2, \eta_3, \ldots, \eta_k$. Let Φ denote the set of distinct scalars among $\eta_2, \eta_3, \ldots, \eta_k$. We observe $U_\eta \neq 0$ if and only if $\eta \in \Phi$. By (3) and since $E_1^* A E_1^*$ is symmetric with real entries we f nd

$$U = \sum_{\eta \in \Phi} U_\eta \qquad \text{(orthogonal direct sum)}. \tag{52}$$

The following result will be useful.

LEMMA 7.2. *With reference to Def nition* 5.1*, let E denote a primitive idempotent of* Γ *and let $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ denote the corresponding dual eigenvalue sequence. Then*

$$|X| E_1^* E E_1^* = (\theta_0^* - \theta_2^*) E_1^* + (\theta_1^* - \theta_2^*) E_1^* A E_1^* + \theta_2^* E_1^* J E_1^*. \tag{53}$$

PROOF. By (10),

$$\begin{aligned} E_1^* J E_1^* &= E_1^* \left(\sum_{i=0}^{D} A_i \right) E_1^* \\ &= E_1^* + E_1^* A_1 E_1^* + E_1^* A_2 E_1^* \end{aligned} \tag{54}$$

in view of (9), (41), and (46). Using (19) we similarly f nd

$$\begin{aligned} |X| E_1^* E E_1^* &= E_1^* \left(\sum_{i=0}^{D} \theta_i^* A_i \right) E_1^* \\ &= \theta_0^* E_1^* + \theta_1^* E_1^* A_1 E_1^* + \theta_2^* E_1^* A_2 E_1^*. \end{aligned} \tag{55}$$

Eliminating $E_1^* A_2 E_1^*$ in (55) using (54) we get (53). □

8. SOME INNER PRODUCTS

In this section we are concerned with the following basis.

LEMMA 8.1. *With reference to Def nition* 5.1*, let v denote a nonzero vector in $E_1^* V$ which is orthogonal to s_1. Then the nonvanishing vectors among*

$$E_1 v, E_2 v, \ldots, E_D v \tag{56}$$

form an orthogonal basis for Mv.

PROOF. Recall $E_0, E_1, \ldots, E_D$ form a basis for M. We assume v is orthogonal to s_1 so $E_0 v = 0$. Now apparently the vectors in (56) span Mv. The vectors in (56) are mutually orthogonal by (18) and the result follows. □

Referring to Lemma 8.1, our next goal is to determine which of $E_1 v, E_2 v, \ldots, E_D v$ is zero. To do this, we compute the square norms of these vectors. We begin with a def nition.

DEFINITION 8.2. With reference to Defi ition 5.1, for all $z \in \mathbb{R} \cup \infty$ we defi e

$$\tilde{z} = \begin{cases} -1 - \frac{b_1}{1+z}, & \text{if } z \neq -1, z \neq \infty \\ \infty & \text{if } z = -1 \\ -1, & \text{if } z = \infty. \end{cases} \tag{57}$$

We observe $\tilde{\tilde{z}} = z$ for all $z \in \mathbb{R} \cup \infty$. By Lemma 2.2 neither of θ_1, θ_D is equal to -1, so $\tilde{\theta}_1 = -1 - b_1(1 + \theta_1)^{-1}$ and $\tilde{\theta}_D = -1 - b_1(1 + \theta_D)^{-1}$. By the data in Lemma 2.2 we have $\tilde{\theta}_1 < -1$ and $\tilde{\theta}_D \geq 0$.

The following theorem is due to Caughman [4, Theorem 5.2]. Because our definition are somewhat different from his we give a full proof.

THEOREM 8.3 ([4, THEOREM 5.2]). *With reference to Def nition* 5.1*, let v denote a nonzero vector in $E_1^* V$ which is orthogonal to s_1. Assume v is an eigenvector for $E_1^* A E_1^*$; let η denote the corresponding eigenvalue.*

(i) *Assume $\eta \neq -1$. Then*

$$\|E_i v\|^2 = \frac{m_i(k - \theta_i)(\tilde{\eta} - \theta_i)}{k|X|(\tilde{\eta} + 1)} \|v\|^2 \qquad (0 \leq i \leq D). \tag{58}$$

(ii) *Assume $\eta = -1$. Then*

$$\|E_i v\|^2 = \frac{m_i(k - \theta_i)}{k|X|} \|v\|^2 \qquad (0 \leq i \leq D). \tag{59}$$

PROOF. Observe $v = E_1^* v$ by construction so $E_i v = E_i E_1^* v$. We may now argue

$$\begin{aligned} \|E_i v\|^2 &= (E_i E_1^* v)^t \overline{E_i E_1^* v} && \text{by (2)} \\ &= v^t E_1^{*t} E_i^t \overline{E}_i \overline{E}_1^* \overline{v} \\ &= v^t E_1^* E_i^2 E_1^* \overline{v} && \text{by (15), (16), (39), (40)} \\ &= v^t E_1^* E_i E_1^* \overline{v} && \text{by (17).} \end{aligned} \tag{60}$$

To evaluate (60) we apply Lemma 7.2. To do this we make some comments. Let $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ denote the dual eigenvalue sequence for E_i. We assume v is orthogonal to s_1 so $Jv = 0$. We already mentioned $E_1^* v = v$. From the construction $E_1^* A E_1^* v = \eta v$. Since each of J, E_1^*, A has real entries and since η is real, we see $J\bar{v} = 0$, $E_1^* \bar{v} = \bar{v}$, and $E_1^* A E_1^* \bar{v} = \eta \bar{v}$. Evaluating (60) using Lemma 7.2 and our above comments, we f nd $|X| \|E_i v\|^2$ is equal to $\|v\|^2$ times

$$\theta_0^* - \theta_2^* + \eta(\theta_1^* - \theta_2^*). \tag{61}$$

Observe (61) is equal to $\theta_0^* - \theta_1^*$ times

$$1 + (1 + \eta) \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_1^*}. \tag{62}$$

Using $\theta_0^* = m_i$ and (25) we f nd $\theta_0^* - \theta_1^* = m_i(k - \theta_i)/k$. First assume $\eta = -1$. Then (62) is equal to 1 and (59) follows. Next assume $\eta \neq -1$. Then $\tilde{\eta} = -1 - b_1(1 + \eta)^{-1}$ by Defi ition 8.2 so $(1 + \eta)(1 + \tilde{\eta}) = -b_1$. Evaluating (62) using this and Corollary 3.5 we routinely obtain (58). □

THEOREM 8.4 ([37, THEOREM 1]). *With reference to Def nitions* 5.1 *and* 7.1*, we have $\tilde{\theta}_1 \leq \eta_i \leq \tilde{\theta}_D$ for $2 \leq i \leq k$.*

PROOF. Let the integer i be given, and abbreviate $\eta = \eta_i$. Let v denote a nonzero vector in U_η. First suppose $\eta < \tilde{\theta}_1$. By Definition 8.2 and since $\tilde{\theta}_1 < -1$ we find $-1 < \tilde{\eta} < \theta_1$. Now $\|E_1 v\|^2 < 0$ by Theorem 8.3, a contradiction. Next suppose $\eta > \tilde{\theta}_D$. By Definition 8.2 and since $\tilde{\theta}_D \geq 0$ we find $\theta_D < \tilde{\eta} < -1$. Now $\|E_D v\|^2 < 0$ by Theorem 8.3, a contradiction. □

Referring to Lemma 8.1, we now determine which of $E_1 v, E_2 v, \ldots, E_D v$ is zero.

LEMMA 8.5 ([4, THEOREM 5.4]). *With reference to Definition* 5.1*, let v denote a nonzero vector in $E_1^* V$ which is orthogonal to s_1. Then (i)–(iii) hold below.*

(i) *The vector $E_0 v$ is zero and each of $E_2 v, E_3 v, \ldots, E_{D-1} v$ is nonzero.*
(ii) *$E_1 v = 0$ if and only if $v \in U_{\tilde{\theta}_1}$.*
(iii) *$E_D v = 0$ if and only if $v \in U_{\tilde{\theta}_D}$.*

PROOF. First suppose there exists an integer $n (1 \leq n \leq D)$ such that $E_n v = 0$. We show $n = 1$ or $n = D$, and that $v \in U_{\tilde{\theta}_n}$. We claim v is an eigenvector for $E_1^* A E_1^*$. To see this, in (53) set $E = E_n$ and apply both sides to v. Using $v = E_1^* v$ and $Jv = 0$ we find

$$0 = (\theta_0^* - \theta_2^*)v + (\theta_1^* - \theta_2^*)E_1^* A E_1^* v,$$

where $\theta_0^*, \theta_1^*, \theta_2^*$ are dual eigenvalues for E_n. Observe $\theta_1^* \neq \theta_2^*$; otherwise $\theta_0^* = \theta_2^*$ by the above line, forcing $\theta_0^* = \theta_1^*$ and contradicting (25). Apparently v is an eigenvector for $E_1^* A E_1^*$, as claimed. Let η denote the corresponding eigenvalue. By assumption $E_n v = 0$ so $\|E_n v\|^2 = 0$. Applying Theorem 8.3 we see $\tilde{\eta} = \theta_n$. Since the tilde map is an involution we have $\eta = \tilde{\theta}_n$. By Theorem 8.4 and since $\theta_0 > \theta_1 > \cdots > \theta_D$ we find $n = 1$ or $n = D$. Apparently $v \in U_{\tilde{\theta}_n}$. To finish the proof, suppose $n = 1$ or $n = D$ and assume $v \in U_{\tilde{\theta}_n}$. We show $E_n v = 0$. Observe v is an eigenvector for $E_1^* A E_1^*$ with eigenvalue $\tilde{\theta}_n$. Applying Theorem 8.3 we find $\|E_n v\|^2 = 0$ so $E_n v = 0$, as desired. The result follows. □

COROLLARY 8.6. *With reference to Definition* 5.1*, let v denote a nonzero vector in $E_1^* V$ which is orthogonal to s_1. Then (i), (ii) hold below.*

(i) *If $v \in U_{\tilde{\theta}_1}$ or $v \in U_{\tilde{\theta}_D}$ then Mv has dimension $D - 1$.*
(ii) *If $v \notin U_{\tilde{\theta}_1}$ and $v \notin U_{\tilde{\theta}_D}$ then Mv has dimension D.*

PROOF. Combine Lemmas 8.1 and 8.5. □

DEFINITION 8.7. With reference to Definition 5.1, let W denote a thin irreducible T-module with endpoint 1. Observe $E_1^* W$ is a one-dimensional eigenspace for $E_1^* A E_1^*$; let η denote the corresponding eigenvalue. We observe $E_1^* W$ is contained in $E_1^* V$ and orthogonal to s_1, so $E_1^* W \subseteq U_\eta$. Apparently $U_\eta \neq 0$ so η is one of $\eta_2, \eta_3, \ldots, \eta_k$. We have $\tilde{\theta}_1 \leq \eta \leq \tilde{\theta}_D$ by Lemma 8.4. We refer to η as the local eigenvalue of W.

With reference to Definition 5.1, let W denote a thin irreducible T-module with endpoint 1 and local eigenvalue η. In order to describe W we distinguish two cases: (i) $\eta = \tilde{\theta}_1$ or $\eta = \tilde{\theta}_D$; (ii) $\tilde{\theta}_1 < \eta < \tilde{\theta}_D$. In this paper we consider case (i). Case (ii) will be considered in a future paper.

9. THE SPACES $U_{\tilde{\theta}_1}$ AND $U_{\tilde{\theta}_D}$

We state our goal for this section. With reference to Definition 5.1, let $n = 1$ or $n = D$ and define $\eta = \tilde{\theta}_n$. We show that for all nonzero $v \in U_\eta$ the space Mv is a thin irreducible T-module with endpoint 1 and local eigenvalue η.

LEMMA 9.1. *With reference to Definition* 5.1*, let* v *denote a vector in* E_1^*V*. Then*

$$E_i^* A_j v = 0 \qquad \text{if } |i-j| > 1 \qquad (0 \le i, j \le D).$$

PROOF. Let i, j be given and assume $|i-j| > 1$. Observe $E_i^* A_j E_1^* = 0$ by (46) so $E_i^* A_j E_1^* v = 0$. Observe $E_1^* v = v$ so $E_i^* A_j v = 0$. □

LEMMA 9.2. *With reference to Definition* 5.1*, let* v *denote a vector in* E_1^*V *which is orthogonal to* s_1*. Then*

$$\sum_{j=0}^{D} E_i^* A_j v = 0 \qquad (0 \le i \le D).$$

PROOF. Observe $Jv = 0$ so $E_i^* J v = 0$. Eliminate J in this expression using (10) to get the result. □

LEMMA 9.3. *With reference to Definition* 5.1*, let* $n = 1$ *or* $n = D$ *and define* $\eta = \tilde{\theta}_n$*. Then for all* $v \in U_\eta$ *we have*

$$\sum_{j=0}^{D} \theta_j^* E_i^* A_j v = 0 \qquad (0 \le i \le D),$$

where $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ *denotes the dual eigenvalue sequence for* θ_n*.*

PROOF. Observe $E_n v = 0$ by Lemma 8.5 so $E_i^* E_n v = 0$. Eliminate E_n in this expression using (19) to get the result. □

LEMMA 9.4. *With reference to Definition* 5.1*, let* $n = 1$ *or* $n = D$ *and define* $\eta = \tilde{\theta}_n$*. Then for all* $v \in U_\eta$ *we have*

$$E_i^* A_i v = \frac{\theta_{i-1}^* - \theta_{i+1}^*}{\theta_{i+1}^* - \theta_i^*} E_i^* A_{i-1} v \qquad (1 \le i \le D-1), \tag{63}$$

$$E_i^* A_{i+1} v = \frac{\theta_{i-1}^* - \theta_i^*}{\theta_i^* - \theta_{i+1}^*} E_i^* A_{i-1} v \qquad (1 \le i \le D-1), \tag{64}$$

where $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ *denotes the dual eigenvalue sequence for* θ_n*. Moreover*

$$E_0^* A_i v = 0, \qquad E_D^* A_i v = 0, \qquad (0 \le i \le D). \tag{65}$$

We note the denominators in (63)*,* (64) *are nonzero by Lemma* 2.1*.*

PROOF. By Lemma 2.1 we find $\theta_{i-1}^* \neq \theta_i^*$ for $1 \le i \le D$. Solving the equations in Lemmas 9.2 and 9.3 using this and Lemma 9.1 we routinely obtain (63)–(65). □

LEMMA 9.5. *With reference to Definition* 5.1*, let* v *denote a vector in* E_1^*V *which is orthogonal to* s_1*. Let the polynomials* $p_0, p_1, \ldots, p_D$ *be from* (26)*. Then*

$$p_i(A)v = E_{i+1}^* A_i v - E_i^* A_{i+1} v \qquad (0 \le i \le D-1). \tag{66}$$

Moreover $p_D(A)v = 0$*.*

PROOF. For $0 \le i \le D-1$ we have

$$\begin{aligned} p_i(A)v &= (A_0 + A_1 + \cdots + A_i)v \\ &= (E_0^* + E_1^* + \cdots + E_D^*)(A_0 + A_1 + \cdots + A_i)v \\ &= \sum E_r^* A_s v, \end{aligned} \tag{67}$$

where the sum is over all integers r, s such that $0 \le r \le i+1$, $0 \le s \le i$, and $|r-s| \le 1$. Cancelling terms in (67) using Lemma 9.2 we obtain (66). Recall $p_D(A) = J$ and $Jv = 0$ so $p_D(A)v = 0$. □

THEOREM 9.6. *With reference to Def nition* 5.1*, let* $n = 1$ *or* $n = D$ *and defin* $\eta = \tilde{\theta}_n$*. Then for all* $v \in U_\eta$ *we have*

$$E^*_{i+1} A_i v = \sum_{h=0}^{i} \frac{\theta^*_h - \theta^*_{h+1}}{\theta^*_i - \theta^*_{i+1}} p_h(A) v \qquad (0 \le i \le D-1), \tag{68}$$

where $\theta^*_0, \theta^*_1, \ldots, \theta^*_D$ *denotes the dual eigenvalue sequence for* θ_n*. Moreover both sides of* (68) *are zero for* $i = D - 1$*. We note the denominators in* (68) *are nonzero by Lemma* 2.1.

PROOF. To verify (68), in the expression on the right eliminate each of $p_0(A)v$, $p_1(A)v$, $\ldots$, $p_i(A)v$ using (66), and simplify the result using (64), (65). We now have (68). For $i = D - 1$ both sides of (68) are zero by the equation on the right in (65). □

LEMMA 9.7. *With reference to Def nition* 5.1*, let* $n = 1$ *or* $n = D$ *and defin* $\eta = \tilde{\theta}_n$*. Then for all nonzero* $v \in U_\eta$ *the vectors*

$$E^*_{i+1} A_i v \qquad (0 \le i \le D-2)$$

form a basis for Mv*.*

PROOF. By Corollary 8.6 the dimension of Mv is $D - 1$. By this and since A generates M we fi d v, Av, $A^2v, \ldots, A^{D-2}v$ form a basis for Mv. For $0 \le i \le D - 1$ let the polynomial g_i be as in Defi ition 4.1, and recall g_i has degree i. Apparently the vectors $g_i(A)v$ $(0 \le i \le D-2)$ form a basis for Mv. By Theorem 9.6 we have $g_i(A)v = E^*_{i+1}A_i v$ for $0 \le i \le D-2$. The result follows. □

THEOREM 9.8. *With reference to Def nition* 5.1*, let* $n = 1$ *or* $n = D$ *and defin* $\eta = \tilde{\theta}_n$*. Then for all nonzero* $v \in U_\eta$ *the space* Mv *is a thin irreducible* T*-module with endpoint* 1 *and local eigenvalue* η*.*

PROOF. We f rst show Mv is a T-module. It is clear Mv is closed under M. By Lemma 9.7 and (41) we f nd Mv is closed under M^*. Recall M and M^* generate T so Mv is a T-module. We show Mv is irreducible. From Lemma 9.7 we f nd v is a basis for E^*_1Mv. In particular E^*_1Mv has dimension 1. Since Mv is a T-module it is a direct sum of irreducible T-modules. It follows there exists an irreducible T-module W' such that $W' \subseteq Mv$ and such that $E^*_1W' \neq 0$. We show $W' = Mv$. Observe $E^*_1W' \subseteq E^*_1Mv$, and we mentioned E^*_1Mv has dimension 1, so $E^*_1W' = E^*_1Mv$. Now apparently $v \in E^*_1W'$. Observe W' is M-invariant, so $Mv \subseteq W'$, and it follows $W' = Mv$. In particular Mv is irreducible. From Lemma 9.7 we fi d E^*_iMv is 0 for $i \in \{0, D\}$ and has dimension 1 for $1 \le i \le D-1$. Apparently Mv is thin with endpoint 1. We mentioned v is a basis for E^*_1Mv. From the construction $v \in U_\eta$ so Mv has local eigenvalue η. □

10. THE THIN IRREDUCIBLE T-MODULES WITH ENDPOINT 1 AND LOCAL EIGENVALUE $\tilde{\theta}_1$ OR $\tilde{\theta}_D$

With reference to Defi ition 5.1, we now describe the thin irreducible T-modules with endpoint 1 and local eigenvalue $\tilde{\theta}_1$ or $\tilde{\theta}_D$.

THEOREM 10.1. *With reference to Def nition* 5.1*, let* W *denote a thin irreducible* T*-module with endpoint* 1 *and local eigenvalue* $\tilde{\theta}_n$ $(n = 1$ *or* $n = D)$*. Let* v *denote a nonzero vector in* E^*_1W*. Then* $W = Mv$*. The vectors*

$$E_i v \qquad (1 \le i \le D, i \neq n) \tag{69}$$

form a basis for W *and* $E_0 v = 0$*,* $E_n v = 0$*.*

PROOF. Observe W is M-invariant and $v \in W$ so $Mv \subseteq W$. Observe $v \in U_{\tilde{\theta}_n}$ by Definition 8.7; combining this with Theorem 9.8 we find Mv is a T-module. Now $W = Mv$ by the irreducibility of W. We mentioned $v \in U_{\tilde{\theta}_n}$; by this and Lemma 8.5 we find each of the vectors in (69) are nonzero. Moreover $E_0 v = 0$ and $E_n v = 0$. Applying Lemma 8.1 we find the vectors in (69) form a basis for Mv. □

THEOREM 10.2. *With reference to Definition* 5.1*, let W denote a thin irreducible T-module with endpoint* 1 *and local eigenvalue $\tilde{\theta}_n$ ($n = 1$ or $n = D$). The vectors in* (69) *are mutually orthogonal and*

$$\|E_i v\|^2 = \frac{m_i(k - \theta_i)(\theta_n - \theta_i)}{k|X|(\theta_n + 1)}\|v\|^2 \qquad (1 \le i \le D, i \ne n). \tag{70}$$

(*The scalar m_i denotes the multiplicity of θ_i.*)

PROOF. The vectors in (69) are mutually orthogonal by (18). To obtain (70) we apply Theorem 8.3. Set $\eta = \tilde{\theta}_n$ and observe $\eta \ne -1$ by Definition 8.2. Now (58) holds and (70) follows. □

THEOREM 10.3. *With reference to Definition* 5.1*, let W denote a thin irreducible T-module with endpoint* 1 *and local eigenvalue $\tilde{\theta}_n$ ($n = 1$ or $n = D$). Let v denote a nonzero vector in $E^*_1 W$. Then the vectors*

$$E^*_{i+1} A_i v \qquad (0 \le i \le D - 2) \tag{71}$$

form a basis for W.

PROOF. Observe $v \in U_{\tilde{\theta}_n}$ by Definition 8.7 and $W = Mv$ by Theorem 10.1. The result now follows in view of Lemma 9.7. □

THEOREM 10.4. *With reference to Definition* 5.1*, let W denote a thin irreducible T-module with endpoint* 1 *and local eigenvalue $\tilde{\theta}_n$ ($n = 1$ or $n = D$). Let v denote a nonzero vector in $E^*_1 W$. Then for* $0 \le i \le D - 2$ *we have*

$$E^*_{i+1} A_i v = \sum_{\substack{1 \le j \le D \\ j \ne n}} g_i(\theta_j) E_j v, \tag{72}$$

where the polynomial g_i is from (31).

PROOF. By Theorem 9.6 we have $E^*_{i+1} A_i v = g_i(A)v$. In this equation, multiply $g_i(A)v$ on the left by I, expand using (14), and simplify the result using $AE_j = \theta_j E_j (0 \le j \le D)$. Observe $E_0 v = 0$ and $E_n v = 0$ by Theorem 10.1. □

THEOREM 10.5. *With reference to Definition* 5.1*, let W denote a thin irreducible T-module with endpoint* 1 *and local eigenvalue $\tilde{\theta}_n$ ($n = 1$ or $n = D$). The vectors in* (71) *are mutually orthogonal and*

$$\|E^*_{i+1} A_i v\|^2 = \frac{k_{i+1} c_{i+1} b_{i+1}}{k b_1} \frac{\theta^*_{i+1} - \theta^*_{i+2}}{\theta^*_i - \theta^*_{i+1}} \frac{\theta^*_0 - \theta^*_1}{\theta^*_1 - \theta^*_2} \|v\|^2 \qquad (0 \le i \le D - 2), \tag{73}$$

where $\theta^*_0, \theta^*_1, \ldots, \theta^*_D$ denotes the dual eigenvalue sequence for θ_n. We remark the denominators in (73) *are nonzero by Lemma* 2.1.

PROOF. The vectors in (71) are mutually orthogonal by (43). To verify (73), in the left-hand side eliminate $E^*_{i+1} A_i v$ using (72), and evaluate the result using Theorems 4.8 and 10.2. □

THEOREM 10.6. *With reference to Definition* 5.1*, let W denote a thin irreducible T-module with endpoint* 1 *and local eigenvalue $\tilde{\theta}_n$ ($n = 1$ or $n = D$). With respect to the basis for W given in* (71) *the matrix representing A is*

$$\begin{pmatrix} \alpha_0 & \beta_1 & & & & \mathbf{0} \\ c_1 & \alpha_1 & \beta_2 & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \beta_{D-2} \\ \mathbf{0} & & & & c_{D-2} & \alpha_{D-2} \end{pmatrix},$$

where the α_i, β_i are from Definition 4.5.

PROOF. Let the polynomials g_i be as in Definition 4.1. Setting $\lambda = A$ in (34) and applying the result to v, we find

$$Ag_i(A)v = c_{i+1}g_{i+1}(A)v + \alpha_i g_i(A)v + \beta_i g_{i-1}(A)v \qquad (0 \le i \le D-2),$$

where $g_{-1} = 0$. The result follows in view of Theorem 9.6. □

In summary we have the following theorem.

THEOREM 10.7. *With reference to Definition* 5.1*, let W denote a thin irreducible T-module with endpoint* 1 *and local eigenvalue $\tilde{\theta}_n$ ($n = 1$ or $n = D$). Then W has dimension $D - 1$. For $0 \le i \le D$, $E_i^* W$ is zero if $i \in \{0, D\}$ and has dimension* 1 *if $i \notin \{0, D\}$. Moreover $E_i W$ is zero if $i \in \{0, n\}$ and has dimension* 1 *if $i \notin \{0, n\}$.*

PROOF. The dimension of W is equal to $D - 1$ by Theorem 10.1. Fix an integer i $(0 \le i \le D)$. From Theorem 10.3 we find $E_i^* W$ is zero if $i \in \{0, D\}$ and has dimension 1 if $i \notin \{0, D\}$. From Theorem 10.1 we find $E_i W$ is zero if $i \in \{0, n\}$ and has dimension 1 if $i \notin \{0, n\}$. □

11. SOME MULTIPLICITIES

With reference to Definition 5.1, let W denote a thin irreducible T-module with endpoint 1 and local eigenvalue $\tilde{\theta}_n$ ($n = 1$ or $n = D$). In this section we consider the multiplicity with which W appears in the standard module V.

THEOREM 11.1. *With reference to Definition* 5.1*, let W denote a thin irreducible T-module with endpoint* 1 *and local eigenvalue $\tilde{\theta}_n$ ($n = 1$ or $n = D$). Let W' denote an irreducible T-module. Then the following* (i)*,* (ii) *are equivalent.*

(i) *W and W' are isomorphic as T-modules.*
(ii) *W' is thin with endpoint* 1 *and local eigenvalue $\tilde{\theta}_n$.*

PROOF. (i) ⇒ (ii) Clear.

(ii) ⇒ (i) We display an isomorphism of T-modules from W to W'. Observe $E_1^* W$ and $E_1^* W'$ are both nonzero. Let v (resp. v') denote a nonzero vector in $E_1^* W$ (resp. in $E_1^* W'$). By Theorem 10.3 the vectors

$$E_{i+1}^* A_i v \qquad (0 \le i \le D-2) \tag{74}$$

form a basis for W. Similarly the vectors

$$E_{i+1}^* A_i v' \qquad (0 \le i \le D-2) \tag{75}$$

form a basis for W'. Let $\sigma : W \to W'$ denote the isomorphism of vector spaces that sends $E^*_{i+1}A_i v$ to $E^*_{i+1}A_i v'$ for $0 \le i \le D-2$. We show σ is an isomorphism of T-modules. By Theorem 10.6 the matrix representing A with respect to the basis (74) is equal to the matrix representing A with respect to the basis (75). It follows $\sigma A - A\sigma$ vanishes on W. From the construction we find that for $0 \le h \le D$, the matrix representing E^*_h with respect to the basis (74) is equal to the matrix representing E^*_h with respect to the basis (75). It follows $\sigma E^*_h - E^*_h\sigma$ vanishes on W. The algebra T is generated by A, $E^*_0, E^*_1, \dots, E^*_D$. It follows $\sigma B - B\sigma$ vanishes on W for all $B \in T$. We now see σ is an isomorphism of T-modules from W to W'. □

LEMMA 11.2. *With reference to Definition* 5.1*, let $n = 1$ or $n = D$ and define $\eta = \tilde{\theta}_n$. Then*

$$U_\eta = E^*_1 H_\eta, \tag{76}$$

where H_η denotes the subspace of V spanned by all the thin irreducible T-modules with endpoint 1 *and local eigenvalue η.*

PROOF. We first show $U_\eta \subseteq E^*_1 H_\eta$. Assume $U_\eta \neq 0$; otherwise the result is trivial. Let v denote a nonzero vector in U_η. By Theorem 9.8 we find Mv is a thin irreducible T-module with endpoint 1 and local eigenvalue η, so $Mv \subseteq H_\eta$. Of course $v \in Mv$ so $v \in H_\eta$. By the construction $v \in E^*_1 V$ so $v = E^*_1 v$. It follows $v \in E^*_1 H_\eta$. We have now shown $U_\eta \subseteq E^*_1 H_\eta$. Next we show $U_\eta \supseteq E^*_1 H_\eta$. To see this observe $E^*_1 H_\eta$ is spanned by the $E^*_1 W$, where W ranges over all thin irreducible T-modules with endpoint 1 and local eigenvalue η. For all such W the space $E^*_1 W$ is contained in U_η by Definition 8.7. It follows $U_\eta \supseteq E^*_1 H_\eta$. □

DEFINITION 11.3. With reference to Definition 5.1, and from our discussion in Section 5, the standard module V can be decomposed into an orthogonal direct sum of irreducible T-modules. Let W denote an irreducible T-module. By the multiplicity with which W appears in V, we mean the number of irreducible T-modules in the above decomposition which are isomorphic to W.

DEFINITION 11.4. With reference to Definition 5.1, let $n = 1$ or $n = D$ and define $\eta = \tilde{\theta}_n$. We let μ_η denote the multiplicity with which W appears in V, where W is a thin irreducible T-module with endpoint 1 and local eigenvalue η. If no such W exists we set $\mu_\eta = 0$.

THEOREM 11.5. *With reference to Definition* 5.1*, let $n = 1$ or $n = D$ and define $\eta = \tilde{\theta}_n$. Then the following scalars* (*i*)–(*iii*) *are equal:*

(i) *The scalar μ_η from Definition* 11.4.
(ii) *The dimension of U_η.*
(iii) *The number of times η appears among $\eta_2, \eta_3, \dots, \eta_k$.*

(The scalars η_i are from Definition 7.1.*)*

PROOF. We mentioned below Definition 7.1 that the above scalars (ii), (iii) are equal. We now show that scalars (i), (ii) are equal. In view of Lemma 11.2 it suffices to show the dimension of $E^*_1 H_\eta$ is μ_η. Observe H_η is a T-module so it is an orthogonal direct sum of irreducible T-modules. More precisely

$$H_\eta = W_1 + W_2 + \cdots + W_m \qquad \text{(orthogonal direct sum)}, \tag{77}$$

where m is a nonnegative integer, and where $W_1, W_2, \dots, W_m$ are thin irreducible T-modules with endpoint 1 and local eigenvalue η. Apparently m is equal to μ_η. We show m is equal to the dimension of $E^*_1 H_\eta$. Applying E^*_1 to (77) we find

$$E^*_1 H_\eta = E^*_1 W_1 + E^*_1 W_2 + \cdots + E^*_1 W_m \qquad \text{(orthogonal direct sum)}. \tag{78}$$

Observe each summand on the right in (78) has dimension 1. These summands are mutually orthogonal so m is equal to the dimension of $E_1^* H_\eta$. Now apparently μ_η is equal to the dimension of $E_1^* H_\eta$, as desired. It follows that scalars (i), (ii) above are equal. □

12. Some Comments on Thin Irreducible T-modules with Endpoint 1

With reference to Defi ition 5.1, it is natural to consider the case in which every irreducible T-module with endpoint 1 is thin with local eigenvalue $\tilde{\theta}_1$ or $\tilde{\theta}_D$. We will consider this case in the next section. In the present section we obtain some preliminary results.

Lemma 12.1. *With reference to Def nition* 5.1*, let W denote a thin irreducible T-module with endpoint* 1 *and local eigenvalue η ($\tilde{\theta}_1 < \eta < \tilde{\theta}_D$). Then W has dimension D. Let v denote a nonzero vector in $E_1^* W$. Then $W = Mv$. The vectors*

$$E_1 v, E_2 v, \ldots, E_D v \tag{79}$$

form a basis for W and $E_0 v = 0$. For $0 \le i \le D$ the space $E_i^* W$ is zero if $i = 0$ and has dimension 1 *if* $1 \le i \le D$*. Moreover $E_i W$ is zero if $i = 0$ and has dimension* 1 *if* $1 \le i \le D$*.*

Proof. Observe $v \in U_\eta$ by Defi ition 8.7. By this and Corollary 8.6 we f nd the dimension of Mv is D. Observe $Mv \subseteq W$ so the dimension of W is at least D. Recall W is the direct sum of the nonzero spaces among $E_0^* W, E_1^* W, \ldots, E_D^* W$. Observe $E_0^* W = 0$ since W has endpoint 1. For $1 \le i \le D$ the dimension of $E_i^* W$ is at most 1 since W is thin. It follows the dimension of W is at most D. From this and our above comments W has dimension D and $W = Mv$. Moreover $E_i^* W$ has dimension 1 for $1 \le i \le D$. Combining Lemmas 8.5 and 8.1 we fin the vectors (79) form a basis for W and $E_0 v = 0$. Apparently for $0 \le i \le D$, $E_i W$ is zero if $i = 0$ and has dimension 1 if $1 \le i \le D$. □

Theorem 12.2. *With reference to Def nition* 5.1*, let W denote a thin irreducible T-module with endpoint* 1 *and local eigenvalue η. Then the following* (*i*), (*ii*) *are equivalent.*

(i) $\eta = \tilde{\theta}_1$ *or* $\eta = \tilde{\theta}_D$.
(ii) $E_D^* A_D E_1^*$ *vanishes on* W.

Proof. (i) ⇒ (ii) By Theorem 10.7 we have $E_D^* W = 0$. The space $A_D E_1^* W$ is contained in W so E_D^* vanishes on $A_D E_1^* W$. Apparently $E_D^* A_D E_1^*$ vanishes on W.

(ii) ⇒ (i) We assume $\eta \ne \tilde{\theta}_1$, $\eta \ne \tilde{\theta}_D$ and get a contradiction. Observe $\tilde{\theta}_1 < \eta < \tilde{\theta}_D$ by Theorem 8.4. Let v denote a nonzero vector in $E_1^* W$. We show $E_D^* A_j v = 0$ for $0 \le j \le D$. By Lemma 9.1 we f nd $E_D^* A_j v = 0$ for $0 \le j \le D - 2$. We assume $E_D^* A_D E_1^*$ vanishes on W; by this and since $v = E_1^* v$ we fi d $E_D^* A_D v = 0$. Observe v is orthogonal to s_1; applying Lemma 9.2 we f nd $\sum_{j=0}^{D} E_D^* A_j v = 0$. Combining these facts we f nd $E_D^* A_{D-1} v = 0$. We have now shown $E_D^* A_j v = 0$ for $0 \le j \le D$. Recall $A_0, A_1, \ldots, A_D$ form a basis for M so $E_D^* M v = 0$. We have $Mv = W$ by Lemma 12.1 so $E_D^* W = 0$. However $E_D^* W \ne 0$ by Lemma 12.1, a contradiction. We conclude $\eta = \tilde{\theta}_1$ or $\eta = \tilde{\theta}_D$. □

Corollary 12.3. *With reference to Def nition* 5.1*, assume Γ has intersection number $a_D = 0$. Let W denote a thin irreducible T-module with endpoint* 1 *and local eigenvalue η. Then $\eta = \tilde{\theta}_1$ or $\eta = \tilde{\theta}_D$.*

Proof. By (45) and since $a_D = 0$ we f nd $E_D^* A_D E_1^* = 0$. Apparently $E_D^* A_D E_1^*$ vanishes on W. Now $\eta = \tilde{\theta}_1$ or $\eta = \tilde{\theta}_D$ by Theorem 12.2. □

13. Tight Graphs

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$, valency k, and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. In [26] Jurišić *et al.* proved that the intersection numbers a_1, b_1 satisfy

$$\left(\theta_1 + \frac{k}{a_1 + 1}\right)\left(\theta_D + \frac{k}{a_1 + 1}\right) \geq -\frac{k a_1 b_1}{(a_1 + 1)^2}. \tag{80}$$

Following [26], we say Γ is *tight* whenever Γ is nonbipartite and equality holds in (80).

In this section we give two characterizations of the tight condition in terms of the subconstituent algebra T. These characterizations involve the thin irreducible T-modules with endpoint 1 and local eigenvalue $\tilde{\theta}_1$ or $\tilde{\theta}_D$. In order to motivate our results we sketch a proof of (80) due to Jurišić and Koolen [23, Theorem 2.2].

Proof of (80). Fix $x \in X$ and let $\eta_1, \eta_2, \ldots, \eta_k$ denote the corresponding local eigenvalues of Γ. Consider the sum

$$\sum_{i=1}^{k} (\eta_i - \tilde{\theta}_1)(\eta_i - \tilde{\theta}_D). \tag{81}$$

We evaluate (81) in two ways. First, by Theorem 8.4 and since $\eta_1 = a_1$ we fi d (81) is at most $(a_1 - \tilde{\theta}_1)(a_1 - \tilde{\theta}_D)$. Second, we determine (81) by computing $\sum_{i=1}^{k} \eta_i$ and $\sum_{i=1}^{k} \eta_i^2$. Let the subgraph Δ and the matrix $\breve{A}$ be as in Definitio 7.1. Each diagonal entry of $\breve{A}$ is zero so the trace of $\breve{A}$ is zero. By defi ition $\eta_1, \eta_2, \ldots, \eta_k$ are the eigenvalues of $\breve{A}$ so $\sum_{i=1}^{k} \eta_i = 0$. Recall Δ is regular with valency a_1 so each diagonal entry of $\breve{A}^2$ is a_1. Apparently the trace of $\breve{A}^2$ is $k a_1$ so $\sum_{i=1}^{k} \eta_i^2 = k a_1$. From these comments the expression (81) is equal to $k a_1 + k \tilde{\theta}_1 \tilde{\theta}_D$. We now have an inequality involving $k, a_1, \tilde{\theta}_1, \tilde{\theta}_D$; eliminating $\tilde{\theta}_1, \tilde{\theta}_D$ using (57) and simplifying we get (80). □

In order to gain some insight into the case in which Γ is tight, we examine the above proof. We begin with a defi ition.

Definition 13.1. With reference to Defi ition 5.1, we say Γ is spectrally tight with respect to x whenever η_i is one of $\tilde{\theta}_1, \tilde{\theta}_D$ for $2 \leq i \leq k$. (The scalars η_i and $\tilde{\theta}_i$ are from Defi itions 7.1 and 8.2, respectively.)

From the above proof of (80) we routinely obtain the following.

Lemma 13.2 ([23, Theorem 2.2]). *Let Γ denote a distance-regular graph with diameter $D \geq 3$. Then the following (i)–(iii) are equivalent.*

(i) Γ *is tight.*
(ii) Γ *is nonbipartite and spectrally tight with respect to each vertex.*
(iii) Γ *is nonbipartite and spectrally tight with respect to at least one vertex.*

We wish to consider Lemma 13.2 from the point of view of the subconstituent algebra. To do this we use the following def nition.

Definition 13.3. With reference to Defi ition 5.1, we say Γ is algebraically tight with respect to x whenever every irreducible T-module with endpoint 1 is thin with local eigenvalue $\tilde{\theta}_1$ or $\tilde{\theta}_D$.

The notions of spectrally tight and algebraically tight are related as follows.

LEMMA 13.4. *With reference to Definition 5.1, the following (i), (ii) are equivalent.*

(i) Γ *is spectrally tight with respect to* x.
(ii) Γ *is algebraically tight with respect to* x.

PROOF. (i) $\Rightarrow$ (ii) Let W denote an irreducible T-module with endpoint 1. We show W is thin with local eigenvalue $\tilde{\theta}_1$ or $\tilde{\theta}_D$. Observe E_1^*W is nonzero and invariant under $E_1^*AE_1^*$. Therefore there exists a nonzero vector $v \in E_1^*W$ which is an eigenvector for $E_1^*AE_1^*$. Let η denote the corresponding eigenvalue. Observe E_1^*W is orthogonal to s_1 so v is orthogonal to s_1. Now η is one of $\eta_2, \eta_3, \ldots, \eta_k$. By this and Definition 13.1 we find η is one of $\tilde{\theta}_1, \tilde{\theta}_D$. By Theorem 9.8 we find Mv is a thin irreducible T-module with endpoint 1 and local eigenvalue η. Observe $Mv \subseteq W$ so $Mv = W$ by the irreducibility of W. Apparently W is thin with local eigenvalue η and the result follows.

(ii) $\Rightarrow$ (i) Let S denote the subspace of V spanned by all the irreducible T-modules with endpoint 1. Then

$$S = H_{\tilde{\theta}_1} + H_{\tilde{\theta}_D} \qquad \text{(orthogonal direct sum)}, \tag{82}$$

where $H_{\tilde{\theta}_1}$ and $H_{\tilde{\theta}_D}$ are from Lemma 11.2. Recall U denotes the orthogonal complement of s_1 in E_1^*V. Applying E_1^* to each term in (82), and evaluating the result using Lemma 11.2 and $E_1^*S = U$, we obtain

$$U = U_{\tilde{\theta}_1} + U_{\tilde{\theta}_D} \qquad \text{(orthogonal direct sum)}. \tag{83}$$

Comparing (83) and (52) we find η_i is one of $\tilde{\theta}_1, \tilde{\theta}_D$ for $2 \le i \le k$. Now Γ is spectrally tight with respect to x by Definition 13.1. $\square$

DEFINITION 13.5. With reference to Definition 5.1, we say Γ is tight with respect to x whenever the equivalent conditions (i), (ii) hold in Lemma 13.4.

Combining Lemma 13.2 and Definition 13.5 we immediately obtain the following theorem.

THEOREM 13.6. *Let* Γ *denote a distance-regular graph with diameter* $D \ge 3$. *Then the following (i)–(iii) are equivalent.*

(i) Γ *is tight.*
(ii) Γ *is nonbipartite and tight with respect to each vertex.*
(iii) Γ *is nonbipartite and tight with respect to at least one vertex.*

In Theorem 13.6 we obtained a characterization of the tight condition. Combining this result with Corollary 12.3 we obtain another characterization of the tight condition. To state the result we use the following notation. With reference to Definition 5.1, we say Γ is 1-*thin with respect to* x whenever every irreducible T-module with endpoint 1 is thin.

THEOREM 13.7. *Let* Γ *denote a distance-regular graph with diameter* $D \ge 3$. *Then the following (i)–(iii) are equivalent.*

(i) Γ *is tight.*
(ii) Γ *is nonbipartite,* $a_D = 0$, *and* Γ *is 1-thin with respect to each vertex.*
(iii) Γ *is nonbipartite,* $a_D = 0$, *and* Γ *is 1-thin with respect to at least one vertex.*

PROOF. (i) $\Rightarrow$ (ii) Γ is nonbipartite by the definition of tight, and $a_D = 0$ by [26, Theorem 10.4]. Let x denote a vertex in X. By Theorem 13.6 we find Γ is tight with respect to x so Γ is 1-thin with respect to x.

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (i) We show Theorem 13.6(iii) holds. By assumption Γ is nonbipartite. We show Γ is tight with respect to at least one vertex. By assumption there exists a vertex with respect to which Γ is 1-thin. Denote this vertex by x and write $T = T(x)$. Let W denote an irreducible T-module with endpoint 1. Observe W is thin; let η denote the local eigenvalue of W. By Corollary 12.3 and since $a_D = 0$ we find $\eta = \tilde{\theta}_1$ or $\eta = \tilde{\theta}_D$. We have now shown every irreducible T-module with endpoint 1 is thin with local eigenvalue $\tilde{\theta}_1$ or $\tilde{\theta}_D$. By Definition 13.3 Γ is algebraically tight with respect to x. By Definition 13.5 Γ is tight with respect to x. We have now shown Theorem 13.6(iii) holds so Γ is tight by that theorem. $\square$

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